PRODYN TECHNOLOGIES
Precision EM Instrumentation

PRODYN APPLICATION NOTE 1195
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THE EXPONENTIAL MODEL OF ELECTROMAGNETIC PULSE

ElectroMagnetic radiation is the transportation of energy through a medium by simultaneous propagation of a time variant electric field and an associated covariant magnetic field. EM radiation can be in the form of a continuous wave (CW) as in radio waves, or a single burst as in an ElectroMagnetic Pulse (EMP).

The magnetic field in a CW is $B(t) = B_p \sin 2\pi ft$, where B_p is the amplitude or maximum value, f is the frequency or the reciprocal of the period T and t is the time. The signal induced in a conducing loop of area A, normal to this field, is directly proportional to the time rate of change, or derivative of the magnetic field intensity (Faraday's Law, $V = dB/dt\ A$). The derivative of B is $2\pi f B_p \cos 2\pi ft$ so it has the same waveform except for amplitude magnification of $2\pi f$ and a phase shift of $\pi/2$. Note that as frequency increases, the amplitude of the derivative of the field intensity increases, so that at high frequencies, even low field intensities give rise to high rates of change, and induction, which is directly proportional to the rate of change, is high. This is what makes radio feasible, and the higher the frequency (the shorter the wavelength, hence the term "short wave radio"), the lower the power required to induce a given signal in a radio antenna at a given distance from the transmitter.

The fields in an EMP are often best represented by exponential, rather than sinusoidal functions. A typical burst might have a magnetic field intensity component described by a double exponential pulse function of the form

$$B(t) = B_o \left(e^{-at/\tau} - e^{-bt/\tau}\right) \tag{1}$$

where:
B_o = initial value of magnetic field source function (not to be confused with B_p)
a = discharging coefficeint
b = charging coefficient, = a+1
τ = time constant of charging source function.

A double exponential pulse with its source functions, derivative and antiderivative is shown in Figure 1. This function is the product of a charging function of the form $1 - e^{-t/\tau}$ and a discharging function of the form $B_o e^{-at/\tau}$. The product has physical meaning in that a source of radiant energy charges the medium at the point under consideration, while the energy radiating away from the point under consideration simultaneously discharges the medium. The time constant τ is to the double exponential pulse what the period T is to the sine wave. It is

It is the basis of all temporal parameters, and all temporal parameters are derived from it The peak time t_p or time of maximum value t_m (the true rise time of the pulse) is the time at which B(t) reaches B_p. These times are indicated in Figure 1. The practical rise time of a pulse is usually taken as the time

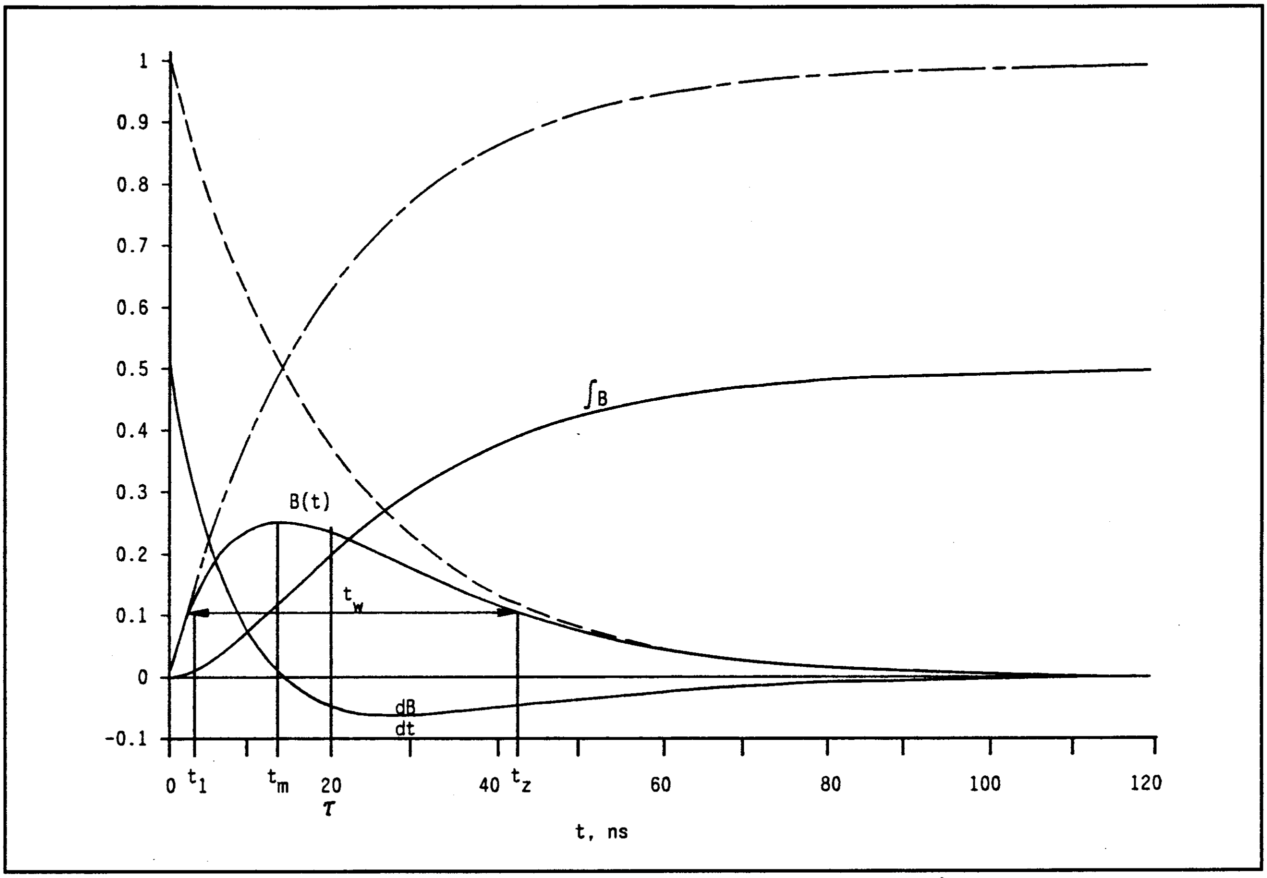

Figure 1. Exponential model of an ElectroMagnetic Pulse.

required for the function to rise from 10 to 90% of its maximum value (more on this later).

The derivative of B is the time rate of change of B:

$$dB/dt = B_o\ /\tau\ \ (-ae^{-at/\tau} + be^{-bt/\tau}) = B_o\ /\tau\ \ (be^{-bt/\tau} - ae^{-at/\tau}) \qquad (2)$$

also shown in Figure 1. This function has the same mathematical form as the function from which it was derived, as is the case with CW, but the constants have a significant effect on the shape of the derived function, whereas with CW, the constants affect only the amplitude and the phase, while the shape remains sinusoidal. Another common feature is that the maximum value of the derivative becomes large as the rise time, which is proportional to the time constant, becomes small. With short rise times (high equivalent frequencies), even low field intensities give rise to high rates of change; and induction, which is directly proportional to the rate of change, is high. This is what makes EMP bursts such a threat to electronic systems, and the shorter the rise time, the lower the power required to induce a given spurious signal in a piece of electronic equipment at a given distance from the source. EM radiation can be measured with instruments which sense the rate of change of the electric or magnetic field. A B-dot sensor (magnetic field sensor) responds to B(t) by sensing dB/dt according to Faraday's law of induction. The transfer function for such a sensor is developed in PRODYN Application Note (PAN) 192, pp 2-3. It is:

$$V_o = A_{eq}\, dB/dt, \tag{3}$$

or in EMP-ese, "Voltage out equals Equivalent Area times B-dot". Thus an EMP with a magnetic field described by B(t) in Figure 1 will induce a signal described by dB/dt in Figure 1 in a B-dot sensor. B(t) is obtained by integrating the output of the sensor.

The time required for the pulse to rise from zero to its maximum value, t_m, is calculated by setting the derivative of B(t) equal to 0:

$$dB/dt = B_o/\tau\ (be^{-bt/\tau} - ae^{-at/\tau}) = 0$$

$$\Rightarrow \quad b/a = e^{tm/\tau} \quad \Rightarrow \quad t_m = \tau \ln(b/a) \tag{4}$$

The maximum value of the pulse is calculated by substituting t_m in equation 1:

$$B_p = B(t_m) = B_o\ (e^{-atm/\tau} - e^{-btm/\tau})$$

$$\Rightarrow \quad B_p = B_o[(b/a)^{-a} - (b/a)^{-b}] \tag{5}$$

or, letting $k_p = B_p/B_o = (b/a)^{-a} - (b/a)^{-b}$, Equation 5 can be written as $B_p = k_p B_o$.

The antiderivative of B is

$$\int B(t) = B_o\ \tau/ab\ (ae^{-bt/\tau} - be^{-at/\tau}), \tag{6}$$

also shown in Figure 1. The antiderivative is a very good representation of step functions which occur in the real world. Many times in EMP diagnostic testing, the subject EMP is really the leading edge of a step function such as $\int B(t)$ in Figure 1. In this instance, the EMP (or more precicely the EM step) will induce a signal described by B(t) in Figure 1 in a B-dot sensor. As before, $\int B(t)$ is obtained by integrating the output of the sensor.

The area under the function B(t), which is proportional to the energy in the pulse, can be calculated by integrating Equation 6 from zero to t:

$$_0\int^t B(t)dt = B_o\ \tau/ab\ (ae^{-bt/\tau} - be^{-at/\tau} + 1)$$

Substituting t = ∞ into the integral gives the total area under the curve:

$$A = {_0\int^\infty} B(t)dt = B_o\ \tau/ab \tag{7}$$

Defining the "width" of the pulse, t_w, as the area under the curve, A, divided by the maximum value, B_p, ie, the area under the curve is the width times the height,

$$t_w = A/B_p = B_o\ \tau/B_p\ ab = \tau/k_p\ ab \tag{8}$$

Now define t_1 as the start and t_2 as the end of the time interval t_w, so that $t_2 = t_1 + t_w$ (see Figure 1). Setting $B(t_1) = B(t_2) = B(t_1 + t_w)$ yields

$$t_1 = \tau\ \ln\ [(e^{-1/kp\ a} -1)\ /\ (e^{-1/kp\ b} -1)]$$

or, with $k_w = (e^{-1/kp\ a} -1)\ /\ (e^{-1/kp\ b} -1)$,

$$t_1 = \tau\ \ln\ k_w \tag{9}$$

Substituting t_1 in Equation (1) gives the value of B at t_1 and t_2. Using $B_p = k_p\ B_o$ leads us to the ratio of the value at t_1 and t_2 to the maximum value, which we call ß:

$$B(t_1)/B_p = k_p(k_w^{-a} - k_w^{-b}) = \beta \tag{10}$$

For ordinary values of a (a >.1), ß(a) rapidly approaches its asymtote, ß' = 0.430820:

a	.1	1	10	100
ß(a)	.3795	.4200	.4306	.4308
dß/ß'	.1191	.0251	.0005	

For most real situations .1 < a < 10, and most often a = 1, which leads to $B(t_1) = .42\ B_p$. The width of this pulse is not taken at the half maximum, but rather at the "42% maximum." Actually, the only common pulse shapes for which t_w satisfies the areal definition and is measured at the half maximum, ie, for which ß = .50, are the sine squared and triangular pulses. The single exponential pulse ($B(t) = e^{-at}$) has ß = .37 and the half sine pulse ($B(t) = \sin 2\pi ft$) has ß = .58. The very common practice of defining the width of all pulses as the "full width half maximum" is more conveinent than correct.

In all the foregoing, the defining parameters B_o , τ, and a have significant mathematical meaning. The author believes that they also have profound physical significance. B_o determines the peak value of B(t), τ sets the time base and a determines the exact shape of the pulse. The parameters can be chosen so as to model an unlimited variety of physical phenomena. The author's research is directed toward developing the physical relationships between the defining parameters of the exponential model and the boundary conditions of real EMPs. The exponential model may be very useful in predicting the threat levels from weapon data, damage potential from meterological data, etc. The model is presently used to postulate EMP data for systems response tests.

An important special case of the exponential model applies when an electromagnetic pulse is propagated through free space. The time constant of the charging source function, τ, is equal to the time constant of the discharging source function, $a\tau$, or a = 1. In this case b = 2, $t_m = .693\tau$, $B_p = B_o/4$, $A = B_o\tau/2 = 2B_p\tau$, $t_w = 2\tau$ and $t_1 = .127\tau$. Figure 1 is an example if this case, with $B_o = 1$ and $\tau = 20$ ns.

Rise Time and Bandwidth (Time and Frequency Domains)

It is most convenient to consider continuous wave (CW) ElectroMagnetic radiation in the frequency domain, and ElectroMagnetic Pulse (EMP) bursts in the time domain. This leads to expressing the frequency range or bandwidth of a measurement system used for CW measurements, while expressing the rise time of a measurement system (perhaps the same system) used for pulse measurements. It is helpful to remember that CW and EMP radiation are two forms of the same phenomenon, and that both forms can be considered in either domain. In other words, a sine wave has a rise time and a pulse has an equivalent frequency. The temporal relationship between the exponential model of an EMP and the sinusoidal model of a CW (see Figure 2) can be derived by equating the initial slope of a double exponential pulse of the form $B(t) = B_o\ (e^{-t/a\tau} - e^{-t/b\tau})$ to the initial slope of a sine wave $B(t) = B_p \sin 2\pi f\ t$. Differentiating both functions with respect to time and equating the slopes at t = 0 yields:

$$B_o/\tau = 2\pi B_p/T \qquad (11)$$

Combining Equation 11 with Equation 4 yields:

$$t_m = \ln(b/a)\ T\ B_o/2\pi B_p$$

Combining this with equation 5 yields:

$$t_m = T\ \ln(b/a)/2\pi[(b/a)-a - (b/a)-b] \qquad (12)$$

or, by recalling that b = a+1 and defining $k_m(a) = \ln((a+1)/a)/2\pi[((a+1)/a)^{-a} - ((a+1)/a)^{-b}]$, we can write

$$t_m = k_m T$$

For ordinary values of a (a >.1), $k_m(a)$ rapidly approaches its asymtote, $k_m' = 0.432627$:

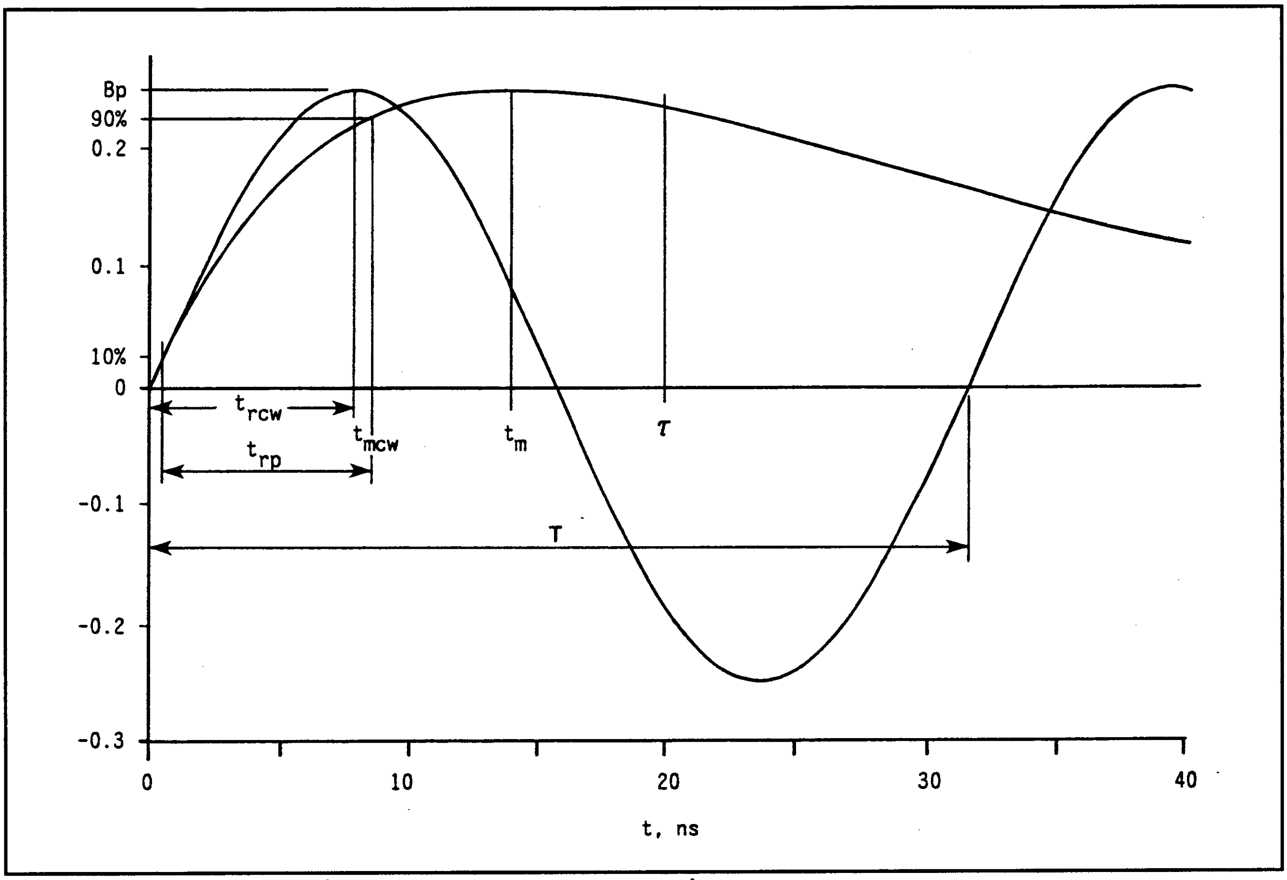

Figure 2. Comparison of pulse and sine wave.

a	.1	1	10	100
$k_m(a)$	.5336	.4413	.4328	.4326
$d\ k_m/k_m'$	.2334	.0200	.0004	

For most real situations .1 < a < 10, and most often a = 1, which leads to $t_m = .44T$. This is the "0 to 100%" rise time of the exponential pulse.

It is natural to compare the 0 to 100% rise time of the exponential pulse to the 0 to 100% rise time of a sine wave, which is $t_{rcw} = T/4 = .25\ T = .25/f$, as shown in Figure 2, but alas! The rise time of the pulse is .44 T = .44/f. This is because the rate of change of the double exponential pulse decreases exponentially as the value increases, which means that the last 10% of the rise takes 40% of the rise time, while the first 10% only takes only 4%. Early in the history of EMP engineering, some bright young star made the practical observation that if we define the rise time of a pulse as the time it takes to get from 10% to 90% of the maximum, then the pulse rise time is 100% - 40%- 4% = 56% of .44 T, which, lo and behold, is .25 T! Regardless of the validity of the premise, and regardless of the fact that the "10 to 90%" rise time of a sine wave is only .16 T, the paradigm was enshrined:

$$f_o = 1/T = .25/t_r \qquad (13)$$

where:

f_o = equivalent frequency of the pulse = frequency of the sine wave
T = period of the sine wave
t_r = 100% rise time of the sine wave and 10 to 90% rise time of the pulse.

This result is used in PAN 890, pp 7-8 to derive the more commonly used expression

$$f_3 = 1/T = .35/t_r \qquad (14)$$

where

f_3 = 3 dB equivalent frequency of the pulse = frequency of the sine wave whose amplitude is 3 dB down from the sine wave of frequency f_o due to frequency response limitations.

Arbritrary Temporal Parameters of the Exponential Model

The "10 to 90%" rise time t_r and the "full width half maximum" pulse width t_{fwhm} or t_H (the time it takes B(t) to go from $B_p/2$ to B_p to $B_p/2$ again) are defined arbitrarily, and are not inherent in the exponential model. We cannot easily solve Equation 1 explicitly for t in terms of selected values of B(t), but we can solve the equations resulting from setting the source functions equal to selected values of B(t). This yields first approximations of the required values of t, which can be refined by successive approximation using Newton's method. For example, to find the value of t for which B(t) = $.9B_p$, we set the charging source function equal to $.9B_p$:

$$1 - e^{-t90'/\tau} = .9B_p$$

$$=> \quad t_{90}' = -\tau \ln(1-.9B_p) \qquad (15)$$

This value is substituted in Equation 1, yielding $B(t_{90}')$ somewhat less than $.9B_p$ (see Figure 3). The difference between $.9B_p$ and t_{90}') is the differential of B, or ΔB. It is used to calculate a time differential as follows:

$$\Delta t = \Delta B/(dB/dt)_{t90'}$$

The time differentail is added to t_{90}' to obtain t_{90}'', and $B(t_{90}'')$, which is still slightly less than $.9B_p$, is calculated. This yields a new ΔB, from which a new Δt is calculated, and so on until $B(t_{90})$ is sufficiently close to $.9B_p$. Three iterations are usually sufficient for t_{90}, which has the worst first approximation. Values of t_{10} and t_{50g} for B(t) = 10 and 50% of B_p are obtained similarly for the growth of B(t), and the value of t_{50d} for B(t) = 50% of B_p for the decay of B(t) is obtained by setting the discharging source function equal to $.5B_p$:

$$B_o e^{-at50d'/\tau} = .5B_p$$

$$\Rightarrow \quad t_{50d}' = -\tau/a \ln(k_p\ /2) \tag{16}$$

The 10 to 90% rise time of the pulse is $t_{90} - t_{10}$ and the "full width half maximum" pulse width is $t_{50d} - t_{50g}$.

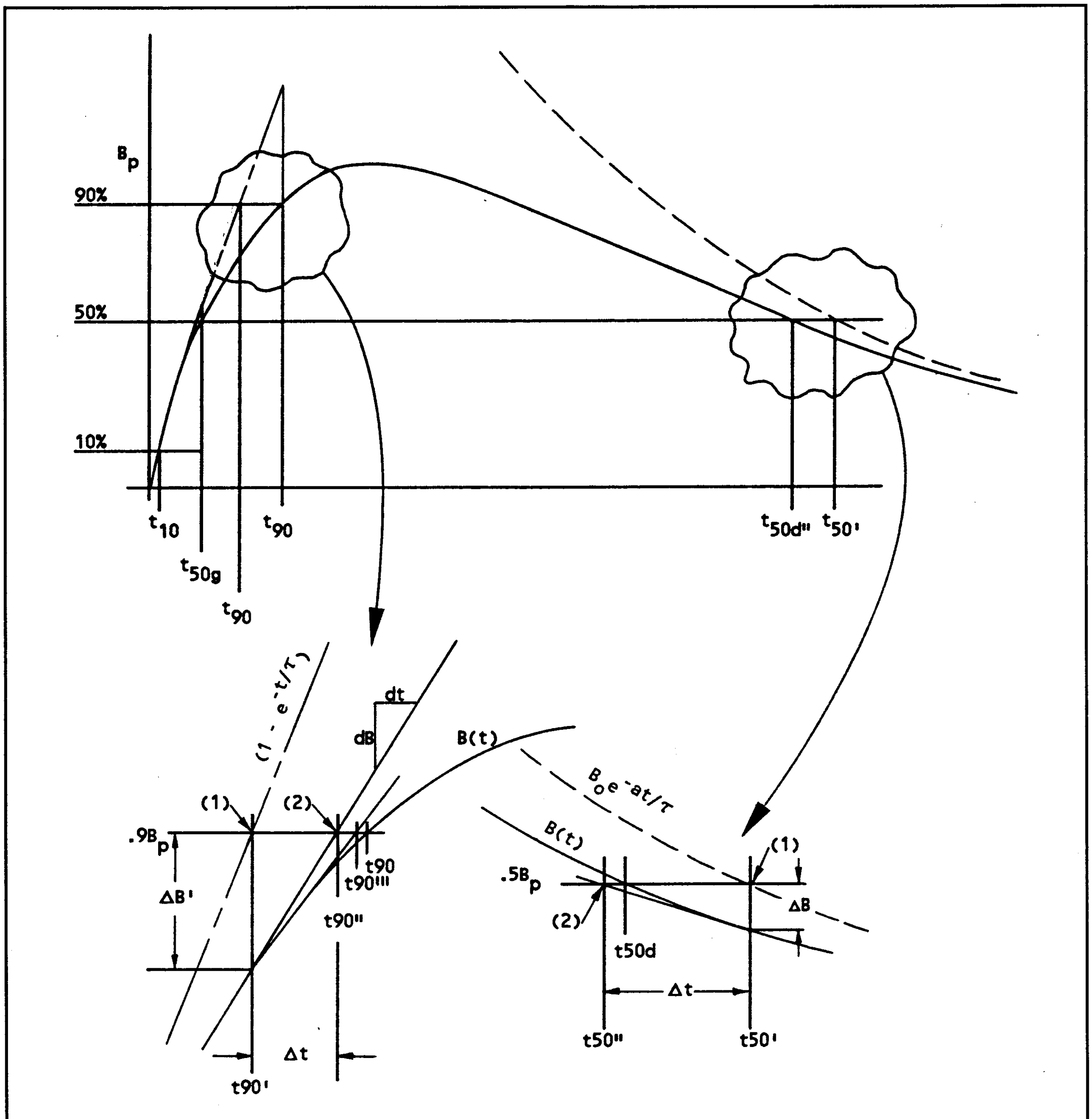

Figure 3. Arbitrary temporal parameters by successive approximation